A SURVEY ON RECENT p-ADIC INTEGRATION THEORIES AND ARITHMETIC APPLICATIONS

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1. Introduction

The aim of the present paper is to give a survey on recent results obtained in [RS] by the authors, where a p-adic integration theory on indefinite quaternion algebras led to the definition of what we call Darmon cycles, a higher weight generalization of Darmon's Stark-Heegner points defined in his 2001 article [D]. At the same time, we emphasize the close analogy with the definite setting pictured in [T1] and explain how the p-adic integration theory on definite quaternion algebras developed there allows to give a purely p-adic analytic description of the p-adic Abel-Jacobi image of so called Heegner cycles. The research was partially conducted in autumn 2009, during the visit at the Centre de Recerca Matemàtica in occasion of the Research Program on Arithmetic Geometry.

Let $S_k(\Gamma_0(N))$ be the $\mathbb{C}$-vector space of even weight $k \geq 2$ cusp forms on $\Gamma_0(N)$ and suppose there is a prime $p \parallel N$. Attached to a new eigenform $f \in S_k(\Gamma_0(N))$ there is a $\mathbb{Q}_p$-adic representation V_f of the global Galois group $G_{\overline{\mathbb{Q}}/\mathbb{Q}}$. Suppose that we have given a quadratic field $K/\mathbb{Q}$ of discriminant D_K prime to N and let H/K be the (narrow) ring class field attached to an order $\mathcal{O}$ of conductor prime to $D_K N$. As in [BK] and [N2], for every place v of K define $H^1_{st}(H_v, V_f)$ to be the kernel of

$$H^1(H_v, V_f) \to \begin{cases} H^1(H_v^{unr}, V_f) & \text{if } v \nmid p \\ H^1(H_v, \mathbf{B}_{st} \otimes_{\mathbb{Q}_p} V_f) & \text{if } v \mid p \end{cases} \tag{1}$$

where H_v^{unr} is the maximal unramified extension of H_v and $\mathbf{B}_{st}$ stands for semistable Fontaine's ring. Define the (semistable) *Mordell-Weil group* of the representation V_f as

$$H_{st}(H, V_f) := \ker\left(H^1(H, V_p) \xrightarrow{\prod \mathrm{res}_v} \prod_v \frac{H^1(H_v, V_p)}{H^1_{st}(H_v, V_p)}\right). \tag{2}$$

As we assume that D_K prime to N, the quadratic field $K/\mathbb{Q}$ induces a factorization $N = pN^+N^-$ as follows: all prime factors of N^+ split in K.and all prime factors of N^- remain inert in K. Assume now that K satisfies the following *Heegner hypothesis*:

- N^- is the squarefree product of an odd (resp. even) number of primes if K is imaginary (resp. real);
- p remains inert in K (write K_p for its p-adic completion).

As we assume that the conductor of $\mathcal{O}$ is prime to $D_K N$, for any character χ: $G_{H/K} \to \mathbb{C}^\times$ of the Galois group of H/K, the root number of the twisted L-series $L(f \otimes K, \chi, s)$ is -1. Since the L-series of $f \otimes H$ admits the factorisation $L(f \otimes H, s) = \prod_{\chi \in G_c^\vee} L(f \otimes K, \chi, s)$, $\mathrm{ord}_{s=k/2} L(f \otimes H, s) \geq |G_{H/K}|$. The Bloch-Beilinson conjecture predicts that $H_{st}(H, V_f)$ has large rank.

When N^- is divisible by an odd (resp. even) number of factors we say that we are in the definite (resp. indefinite setting). Let i_{N^-} be equal to 0 (resp. 1) in the definite (resp. indefinite) case. Recall the Bruhat-Tits tree $\mathcal{T}$ at p, whose definition is recalled in §2, and let $\mathcal{E}$ be the set of oriented edges of $\mathcal{T}$. For any even integer $n \geq 0$ and every $\mathbb{Q}$-algebra

F, let $\mathbf{P}_n(F)$ be the F-vector space of polynomials of degree $\leq n$ with F-coefficients. It can be endowed with a right action of $\mathrm{GL}_2(F)$ by the rule

$$(3) \qquad P(x)\cdot\gamma := \frac{(cx+d)^n}{\det(\gamma)^{n/2}}\cdot P\left(\frac{ax+b}{cx+d}\right), \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix},\ P \in \mathbf{P}_n(F).$$

This way, $\mathbf{V}_n(F) = \mathbf{P}_n^\vee(F) := Hom_F(\mathbf{P}_n(F), F)$, the dual of $\mathbf{P}_n(F)$, inherits a left $\mathrm{GL}_2(F)$-action, which actually descends to $\mathrm{PGL}_2(F)$.

When we are in the definite (resp. indefinite) setting, by a particular case of the Jacquet-Langlands correspondence, there is a modular form on the definite (resp. indefinite) quaternion algebra $\mathcal{B}$ of discriminant $N^-\infty$ (resp. N^-) which is associated to f, unique up to a non-zero scalar factor. More precisely note that, as we assume $p \nmid N^-$, we may identify $\mathcal{B}\otimes_{\mathbb{Q}}\mathbb{Q}_p = \mathrm{M}_2(\mathbb{Q}_p)$. We define in §2 an arithmetic subgroup $\Gamma \subset \mathcal{R}^\times$, where $\mathcal{R}\subset\mathcal{B}$ is a level N^+ Eichler $\mathbb{Z}[1/p]$-order in $\mathcal{B}$. By means of the identification $\mathcal{B}\otimes_{\mathbb{Q}}\mathbb{Q}_p = \mathrm{M}_2(\mathbb{Q}_p)$, Γ acts on the Bruhat-Tits tree as well as on the space $C_{har}(\mathcal{E}, \mathbf{V}_n(K_p))$ of $\mathbf{V}_n(K_p)$-valued harmonic cocyles defined in §2. In particular, we may form the cohomology group

$$\mathbf{H}(K_p) = \mathbf{H}_{N^+}(K_p) := H^{i_{N^-}}(\Gamma, C_{har}(\mathcal{E}, \mathbf{V}_n(K))).$$

As explained in §2, it is naturally endowed with an action of an Hecke algebra. Following [RS] we also recall in §2.2 that, in the indefinite setting, there is an involution W_∞ acting on $\mathbf{H}(K_p)$ and that $\mathbf{H}(K_p)$ factors has the direct sum of an Eisenstein and a cuspidal part. Set $n := k-2$, a notation to be in force for the rest of the paper. As explained in §2.3, the Jacquet-Langlands correspondence mentioned above attaches to f and eigenvector in $\mathbf{H}(K_p)_c^{w_\infty}$ (and $\mathbf{H}(K_p)_c^{w_\infty} = \mathbf{H}(K_p)$ in the definite setting).

Following the construction of [RS] in the indefinite setting, we recall how to construct a monodromy module $\mathbf{D} = \mathbf{D}_{N^-}$ over $\mathbb{Q}_p$attached to the space of pN^--new modular forms. This is done in §4.2. Let $\mathcal{H}_p$ be the p-adic upper halfplane and write $\mathrm{Div}(\mathcal{H}_p)$ to denote the space of divisors supported on $\mathcal{H}_p$ that are fixed by the $G_{\overline{\mathbb{Q}}_p/K_p}$. For the rest of the paper we set $m := n/2$. There is also a p-adic Abel-Jacobi map

$$\Phi^{AJ}\colon H_{i_{N^-}}(\Gamma, \mathrm{Div}(\mathcal{H}_p)\otimes\mathbf{P}_n(K_p)) \to \mathbf{D}_{N^-,K_p}/F^{m+1}\mathbf{D}_{N^-,K_p}.$$

where $\mathbf{D}_{N^-,K_p}$ is the base change of $\mathbf{D}$ (over $\mathbb{Q}_p$) to K_p.

As briefly recalled in §4.1.1 (definite setting) and §4.1.1 (indefinite setting), the $\mathbb{Q}_p$-adic representation V_f appears, respectively, in the p-adic étale cohomology (of the compactification) of a Kuga-Sato variety over the indefinite Shimura curve X_{N^+,pN^-} (definite setting) or X_{pN^+,N^-} (indefinite setting) attached to the indefinite quaterion algebra of discriminant pN^- or N^- and a level N^+ structure of Γ_0-type. The f-isotypic component $\mathbf{D}_f$ of $\mathbf{D}$ is identified with the monodromy module $\mathbb{D}_f = \mathbb{D}_{st}(V_f)$ attached to the modular form f.

Let $\Gamma\backslash\mathrm{Emb}(\mathcal{O},\mathcal{R})$ be set of Γ-equivalence classes of optimal embeddings of $\mathcal{O}$ into $\mathcal{R}$. Attached to $\Psi\in\Gamma\backslash\mathrm{Emb}(\mathcal{O},\mathcal{R})$ there is a naive Heegner cycle (resp. Darmon cycle) $y_\Psi \in H_{i_{N^-}}(\Gamma, \mathrm{Div}(\mathcal{H}_p)\otimes\mathbf{P}_n(K_p))$ when we are in the definite (resp. indefinite) setting.

Suppose that K is imaginary, so that we are in the definite setting. There is also a classical Heegner cycles $y_\Psi^{He} \in CH^{m+1}(\mathcal{M}_{n,H})$, the Chow group of the motive attached to the weight k modular forms on X_{N^+,pN^-} (see §4.1.1 and §5.1). As explained in Theorem 20, the local component at p of the image in $H_{st}(H, V_f)$ of y_Ψ^{He} under the classical Abel-Jacobi map recalled in §5.1 is controlled by $\Phi^{AJ}(y_\Psi)$, which has indeed a global nature. Together with Kolyvagin's type results as in [N1], this partially explain the fact that $H_{st}(H, V_f)$ should have large rank. Finally, guided by the analogies with the definite setting, section §5.2 recalls the conjectures formulated in [RS] when K is real, and gives references for the theoretical evidences. Of course, these conjectures are also motivated by the search for the presence of a large rank in $H_{st}(H, V_f)$.

2. The cohomology of arithmetic groups

Let $\mathcal{B}$ be a definite or indefinite quaternion $\mathbb{Q}$-algebra of discriminant, respectively, $N^-\infty$ or N^-. For a place v let $\mathbb{H}_v$ (resp. $\mathbb{M}_2(\mathbb{Q}_v)$) be the unique (up to isomorphism) division (resp. split) quaternion $\mathbb{Q}_v$-algebra and fix, once and for all, identifications $\iota_v : \mathcal{B}_v = \mathbb{H}_v$ for $v \mid N^-$, $\iota_v : \mathcal{B}_v = \mathbb{M}_2(\mathbb{Q}_v)$ for $v \nmid N^-\infty$, where $\mathcal{B}_v := \mathcal{B} \otimes \mathbb{Q}_v$. If $g \in \mathbb{M}_2(\mathbb{Q}_v)$, we define a_g, b_g, c_g and d_g by $g = \begin{pmatrix} a_g & b_g \\ c_g & d_d \end{pmatrix}$. Let $\mathcal{B}_{\infty,+}^\times \subset \mathcal{B}_\infty^\times$ be the subgroup of elements having positive reduced norm, so that $\mathcal{B}_{\infty,+}^\times = \mathcal{B}_\infty^\times$ in the definite case.

If $v = l \mid N^-$ let $\mathcal{O}_{\mathbb{H}_l}$ be the ring of integers of $\mathbb{H}_v$. If $v = l \nmid N^-\infty$ is a finite prime, we write $\mathcal{R}_{l^n}(\mathbb{Z}_l)$ to denote the standard Eichler $\mathbb{Z}_l$-order of level l^n,

$$\mathcal{R}_0(l^n\mathbb{Z}_l) = \{g \in \mathbb{M}_2(\mathbb{Z}_l) : c_g \equiv 0 \mod l^n\}.$$

We also define the semigroup

$$\Sigma_0(l^n\mathbb{Z}_l) := \left\{g \in \mathbb{M}_2(\mathbb{Z}_l) : c_g \equiv 0 \mod l^n, a_g \in \mathbb{Z}_l^\times, \det(g) \neq 0\right\},$$

so that $\Sigma_0(l^n\mathbb{Z}_l) \subset \mathcal{R}_{l^n}(\mathbb{Z}_l)$. Let $\Gamma_0(l^n\mathbb{Z}_l) := \Sigma_0(l^n\mathbb{Z}_l) \cap \mathbb{GL}_2(\mathbb{Z}_p)$.

For an integer M prime to N^- define the Eichler $\mathbb{Z}$-order of level M in $\mathcal{B}$ by the formula

$$\mathcal{R}_0(M) = \mathcal{R}_0^{N^-}(M) := \mathcal{B}^\times \cap \prod_{l \mid N^-} \iota_l^{-1}(\mathcal{O}_{\mathbb{H}_l}) \prod_{l^n \| M \text{ finite}} \iota_l^{-1}(\mathcal{R}_{l^n}(\mathbb{Z}_l)).$$

We also define the semigroup $\Sigma_0(M)$ and the group $\Gamma_0(M)$ as

$$\Sigma_0(M) := \mathcal{B}^\times \cap \prod_{l \mid N^-} \iota_l^{-1}(\mathcal{O}_{\mathbb{H}_l}) \prod_{l^n \| M \text{ finite}} \iota_l^{-1}(\Sigma_0(l^n\mathbb{Z}_l)),$$
$$\Gamma_0(M) := \mathcal{B}^\times \cap \mathcal{B}_{\infty,+}^\times \prod_{l \mid N^-} \iota_l^{-1}\left(\mathcal{O}_{\mathbb{H}_l}^\times\right) \prod_{l^n \| M \text{ finite}} \iota_l^{-1}(\Gamma_0(l^n\mathbb{Z}_l)),$$

so that $\Gamma_0(M) \subset \Sigma_0(M) \subset \mathcal{R}_0(M)$. Note that $\Gamma_0(M) = \mathcal{R}_0(M)_1^\times$, the subgroup of norm one elements in $\mathcal{R}_0(M)^\times$. Then $(\Gamma_0(M), \Sigma_0(M))$ is an Hecke pair and we may consider the associated Hecke $\mathbb{Q}$-algebra $\mathbb{T}(M) = \mathbb{T}^{N^-}(M) := \mathcal{H}(\Gamma_0(M), \Sigma_0(M))$.

We are concerned with the choice $M = N^+$ or pN^+. We also set

$$\mathcal{R} := \mathcal{R}_0^{N^-}\left(N^+\right)[1/p] = \mathcal{B}^\times \cap \prod_{l \mid N^-} \iota_l^{-1}(\mathcal{O}_{\mathbb{H}_l}) \prod_{p \neq l^n \| N^+ \text{ finite}} \iota_l^{-1}(\mathcal{R}_{l^n}(\mathbb{Z}_l)),$$
$$\Sigma := \mathcal{B}^\times \cap \prod_{l \mid N^-} \iota_l^{-1}(\mathcal{O}_{\mathbb{H}_l}) \prod_{p \neq l^n \| M \text{ finite}} \iota_l^{-1}(\Sigma_0(l^n\mathbb{Z}_l)),$$
$$\Gamma := \mathcal{B}^\times \cap \mathcal{B}_{\infty,+}^\times \prod_{l \mid N^-} \iota_l^{-1}\left(\mathcal{O}_{\mathbb{H}_l}^\times\right) \prod_{p \neq l^n \| M \text{ finite}} \iota_l^{-1}(\Gamma_0(l^n\mathbb{Z}_l)).$$

Note that $\Gamma := \mathcal{R}_1^\times$ is the subgroup of norm one elements in $\mathcal{R}^\times$. Again (Γ, Σ) is an Hecke pair and we may consider the associated Hecke algebra $\mathbb{T}_{1/p}(N^+) = \mathbb{T}_{1/p}^{N^-}(N^+) := \mathcal{H}(\Gamma, \Sigma)$.

For $\mathcal{H} = \mathbb{T}(pN^+)$, $\mathbb{T}(N^+)$ or $\mathbb{T}_{1/p}(N^+)$ and a rational prime $l \nmid N$, there is a Hecke operator T_l attached to the double coset of a norm l element in $\Sigma_0(pN^+)$, $\Sigma_0(N^+)$ or Σ.

Definition 1. *A $\mathcal{H}$-module M admits an* Eisenstein/Cuspidal decomposition *(of weight k) whenever there exists a Hecke operator T_l for some $l \nmid N$ such that $M = M_e \oplus M_c$ and $t_l := T_l - l^{k-1} - 1$ vanishes on M_e and is invertible on M_c.*

In the rest of the paper, whenever M admits an Eisenstein/Cuspidal decomposition, we write M_c to denote its cuspidal part. In the definite case, we always have $M = M_c$ for the $\mathcal{H}$-modules considered in the sequel, with the only possible exception of the case $M = H_0(\Gamma, K_p)$ considered in §4.3. This is also true in the indefinite setting when $k \neq 2$ and $N^- \neq 1$.

Let us remark that, by means of our fixed identification $\mathcal{B}_p = \mathrm{M}_2(\mathbb{Q}_p)$, Γ acts on the p-adic upper halfplane $\mathcal{H}_p$ as well as the Bruhat-Tits' tree whose definition is recalled below. When $\mathcal{B}$ is indefinite, there is also an action of $\Gamma_0(M)$ on the complex upper half plane $\mathcal{H}$ obtained by means of $\mathcal{B}_\infty = \mathrm{M}_2(\mathbb{R})$. The formalism of double cosets endows the cohomology groups $H^{\cdot}(G, A)$ of a Σ_G-module A with a $\mathcal{H}$-module structure for $G = \Gamma_0(pN^+)$, $\Gamma_0(N^+)$ or Γ and $\Sigma_G = \Sigma_0(pN^+)$, $\Sigma_0(N^+)$ or Σ.

Let $\mathcal{T}$ denote Bruhat-Tits' tree attached to $\mathrm{PGL}_2(\mathbb{Q}_p)$, whose set $\mathcal{V}$ of vertices is the set of homothety classes of rank two $\mathbb{Z}_p$-submodules of $\mathbb{Q}_p^2$. Write $\mathcal{E}$ for the set of oriented edges of the tree. Given $e \in \mathcal{E}$, write $s(e)$ and $t(e)$ for the source and target of the edge, and $\bar{e}$ for the edge in $\mathcal{E}$ such that $s(\bar{e}) = t(e)$ and $t(\bar{e}) = s(e)$. Cf. e.g. [DT, §1.3.1] for more details.

Write v_*, $\widehat{v}_*$ for the vertices associated with the standard lattice $L_* := \mathbb{Z}_p \times \mathbb{Z}_p$ and the lattice $\widehat{L}_* := \mathbb{Z}_p \times p\mathbb{Z}_p$, respectively. Let e_* be the edge with source $s(e_*) = v_*$ and $t(e_*) = \hat{v}_*$. Let $\mathcal{V}^+$ (resp. $\mathcal{V}^-$) denote the subset of vertices $v \in \mathcal{V}$ which lie at *even* (resp. *odd*) distance from v_*. Similarly, write $\mathcal{E}^+$ (resp. $\mathcal{E}^-$) for the subset of edges e in $\mathcal{E}$ such that $s(e) \in \mathcal{V}^+$ (resp. $\mathcal{V}^-$).

Let G be a subgroup of $\mathrm{GL}_2(\mathbb{Q}_p)$ and let A be any left G-module. For any set $\mathcal{S}$, e.g. $\mathcal{S} = \mathcal{V}$ or $\mathcal{E}$, write $C(\mathcal{S}, A)$ for the group of functions on $\mathcal{S}$ with values in A, with left action of G by the rule $({}^{\gamma}c)(e) := \gamma(c(\gamma^{-1}e))$. The group $C_{har}(\mathcal{E}, A)$ of A-valued harmonic cocyles is defined by the following exact sequence (cf. [G, Lemma 24] for right exactness):

$$(4)\qquad \begin{array}{ccccccccc} 0 & \to & C_{har}(\mathcal{E}, A) & \to & C_0(\mathcal{E}, A) & \stackrel{\varphi}{\to} & C(\mathcal{V}, A) & \to & 0 \\ & & & & \varphi(c)(v) & := & \sum_{s(e)=v} c(e). & & \end{array}$$

We may also consider the following diagram with exact rows, where the second line is obtained from $(\partial^* c)(e) := c(s(e)) - c(t(e))$ and the first line is the pull-back of the second via the inclusion $C_{har}(\mathcal{E}, \mathbf{V}_n) \subset C_0(\mathcal{E}, \mathbf{V}_n)$:

$$(5)\qquad \begin{array}{ccccccccc} 0 & \to & A & \to & C_{har}(\mathcal{V}, A) & \stackrel{\partial^*}{\to} & C_{har}(\mathcal{E}, A) & \to & 0 \\ & & \| & & \cap & & \cap & & \\ 0 & \to & A & \to & C(\mathcal{V}, A) & \stackrel{\partial^*}{\to} & C_0(\mathcal{E}, A) & \to & 0. \end{array}$$

The above formalism notably applies with $G = \Gamma \subset \mathrm{GL}_2(\mathbb{Q}_p)$ and $A = \mathbf{V}_n(F)$ as in (3), where F is any $\mathbb{Q}$-algebra such that $\mathcal{B} \otimes_{\mathbb{Q}} F = \mathrm{M}_2(F)$ that we use to let Γ act on $\mathbf{V}_n(F)$.

Until §5, we let $K_p/\mathbb{Q}_p$ be any finite extension of $\mathbb{Q}_p$, unless otherwise stated. From §5 we let K_p be the p-adic completion of the quadratic field considered in the introduction.

2.1. **Definite setting.** Suppose in this subsection that $\mathcal{B}$ is definite. Let $S_k(\Gamma, K_p) = S_k^{N^-}(\Gamma, K_p)$ be the space of weight k cusp forms on the Shimura curve $\Gamma \backslash \mathcal{H}_p$ that are defined over K_p. Given $f \in M_k(\Gamma, K_p)$ write ω_f to denote the associated $\mathbf{V}_n(K_p)$-valued differential form, i.e. $\omega_f(z) := \sum_{i=0}^n z^i f(z)\, dz \otimes \partial^i$ where $\partial^i(X^j) = \delta_j^i$. We may define

$$\begin{aligned} & I colon S_k(\Gamma, K_p) \to H^0(\Gamma, C_{har}(\mathcal{E}, \mathbf{V}_n(K_p))) \\ & I(f)(e) := \mathrm{Res}_e(\omega_f) \in \mathbf{V}_n(K_p). \end{aligned}$$

Since Γ is arithmetic, this is indeed and isomorphism, called p-adic Eichler-Shimura isomorphism (see [dS, Sec. 3]).

The following theorem is proved in [dS, Sec. 3].

Proposition 2. *The boundary map arising from the first line of* (5) *with* $A = \mathbf{V}_n(K_p)$ *is an Hecke equivariant isomorphism*

$$\delta colon H^0(\Gamma, C_{har}(\mathcal{E}, \mathbf{V}_n(K_p))) \xrightarrow{\simeq} H^1(\Gamma, \mathbf{V}_n(K_p)).$$

2.2. **Indefinite setting.** Suppose in this subsection that $\mathcal{B}$ is indefinite. Let $S_k(\Gamma_0(pN^+), \mathbb{C}) = S_k^{N^-}(\Gamma_0(pN^+), \mathbb{C})$ be the space of weight k cusp forms on the Shimura curve $\overline{\Gamma_0(pN^+)\backslash\mathcal{H}}$, where $\overline{(-)}$ is the compactification obtained by adding the cusps when $N^- = 1$. Let $W_\infty \in \mathcal{H}_0(pN^+)$ be the involution obtained from a norm -1 element normalizing $\Gamma_0(pN^-)$. If we have given a $\mathbb{Q}[W_\infty]$ module H, we may write $H = H^+ \oplus H^-$ where H^{w_∞} is the subspace where $W_\infty = w_\infty \in \{\pm 1\}$. As it is well known, the group $H^1(\Gamma, \mathbf{V}_n(\mathbb{C}))$ has an Eisenstein cuspidal decomposition and there is an Eichler-Shimura isomorphism

$$S_k\left(\Gamma_0\left(pN^+\right), \mathbb{C}\right) \to H^1(\Gamma, \mathbf{V}_n(\mathbb{C}))_c^{w_\infty}.$$

As in [RS, §2.3], one may define rational structures on $H^1(\Gamma, \mathbf{V}_n(\mathbb{C}))$ that are preserved by the Hecke algebra $\mathbb{T}(pN^+)$. Indeed, there is a $\mathcal{B}^\times$-representations $\mathbb{V}_n$ over $\mathbb{Q}$ such that $\mathbb{V}_n \otimes_{\mathbb{Q}} F = \mathbf{V}_n(F)$ as $(\mathcal{B} \otimes_{\mathbb{Q}} F)^\times = \mathrm{GL}_2(F)$-modules, whenever $F/\mathbb{Q}$ is a $\mathbb{Q}$-algebra that splits $\mathcal{B}$. The identification depends on the choice of $\mathcal{B} \otimes_{\mathbb{Q}} F = \mathrm{M}_2(F)$. In particular, there are Hecke equivariant indentifications

$$H^1(\Gamma, \mathbb{V}_n)_c^{w_\infty} \otimes_{\mathbb{Q}} \mathbb{C} = H^1(\Gamma, \mathbf{V}_n(\mathbb{C}))_c^{w_\infty},\ H^1(\Gamma, \mathbb{V}_n)_c^{w_\infty} \otimes_{\mathbb{Q}} K_p = H^1(\Gamma, \mathbf{V}_n(K_p))_c^{w_\infty},$$

that we may use to "transport" eigenvectors occurring on $H^1(\Gamma, \mathbf{V}_n(\mathbb{C}))_c^{w_\infty}$ to Hecke eigenvector occurring on $H^1(\Gamma, \mathbf{V}_n(K_p))_c^{w_\infty}$. This motivates our interest in the cohomology groups $H^1(\Gamma, \mathbf{V}_n(K_p))$. As explained in [RS, §2.4-5], the cohomology groups $H^{\cdot}(\Gamma, C_{har}(\mathcal{E}, \mathbb{V}_n))$ admit an Eisenstein/cuspidal decomposition. Indeed, the inclusion arising from (4) induces an identification (see [RS, Lemmas 2.8])

$$H^1(\Gamma, C_{har}(\mathcal{E}, \mathbb{V}_n))_c \xrightarrow{\simeq} H^1(\Gamma, \mathbb{V}_n))_c^{p-new},$$

where $H^1(\Gamma, \mathbb{V}_n))_c^{p-new}$ is the p-new part of $H^1(\Gamma, \mathbb{V}_n)_c$. The following proposition is the analogue of Proposition 2 in this indefinite setting.

Proposition 3. *The boundary map arising from the first line of* (5) *with* $A = \mathbf{V}_n(K_p)$ *induces an Hecke equivariant isomorphism*

$$\delta_c \colon H^1(\Gamma, C_{har}(\mathcal{E}, \mathbf{V}_n(K_p)))_c \xrightarrow{\simeq} H^2(\Gamma, \mathbf{V}_n(K_p))_c.$$

Proof. This is [RS, Lemmas 2.8-9]. □

2.3. **Jacquet-Langlands correspondence.** We end this section by recalling how the above spaces $S_k^{N-}(\Gamma, K_p)$ (definite case) and $S_k^{N^-}(\Gamma_0(pN^+), \mathbb{C})^{p-new}$ (indefinite case) are related to the classical space $S_k^1(\Gamma_0(N), \mathbb{C})^{p-new}$. Let $S_k^1(\Gamma_0(N), \mathbb{C})^{pN^- - new}$ be the subspace of pN^--new modular forms. In the definite case, extends the scalars to $K_p = \mathbb{C}_p$ and fix an identification $\mathbb{C}_p \simeq \mathbb{C}$. By a special case of the Jacquet-Langlands correspondence, there are Hecke equivariant identifications

$$S_k^{N-}(\Gamma, \mathbb{C}_p) \leftrightarrow S_k^1(\Gamma_0(N), \mathbb{C})^{pN^- - new} \text{ (definite case)},$$
$$S_k^{N^-}\left(\Gamma_0\left(pN^+\right), \mathbb{C}\right)^{p-new} \leftrightarrow S_k^1(\Gamma_0(N), \mathbb{C})^{pN^- - new} \text{ (indefinite case)}.$$

Set $\mathbf{H}(K_p) := H^{i_{N^-}}(\Gamma, C_{har}(\mathcal{E}, \mathbf{V}_n(K_p)))$. This motivates our interest in the space $\mathbf{H}(K_p)$, or rather $\mathbf{H}(K_p)_c^*$, where $* = \phi$ or w_∞ according to whenever we are in the definite or indefinite setting. Indeed, according to the discussion in §2.1 and §2.2, it affords the same Hecke eigenvectors occurring in $S_k^1(\Gamma_0(N), \mathbb{C})^{pN^- - new}$ (eventually extending K_p). More generally, if we write $\mathbb{T} := \mathbb{T}(\mathbf{H}_c)$ for the Hecke $\mathbb{Q}$-algebra $\mathbb{T}_{1/p}(N^+)$ acting on $\mathbf{H}(K_p)_c^*$ and $\mathbb{T}^1(N)^{pN^- - new}$ for the Hecke $\mathbb{Q}$-algebra $\mathbb{T}^1(N)$ acting on $S_k^1(\Gamma_0(N), \mathbb{C})^{pN^- - new}$,

$$\mathbb{T} \simeq \mathbb{T}^1(N)^{pN^- - new}. \tag{6}$$

3. The cohomology of distributions and harmonic cocyles

3.1. **The harmonic cocycles and the bounded harmonic cocyles.** The nature of the group Γ heavily depends on whenever we are in the definite or indefinite setting. The following lemma is a well known application of the strong approximation theorem.

Lemma 4. *In the definite setting, $\mathcal{G} := \Gamma\backslash\mathcal{T}$ is finite graph and the stabilizer in Γ of $e \in \mathcal{E}$ (resp. $v \in \mathcal{V}$) is finite.*

In the indefinite setting, $\mathcal{G} := \Gamma\backslash\mathcal{T}$ has two distinct vertices represented by v_* and $\widehat{v}_*$, connected by the edges represented by e_* and its opposite $\overline{e}_*$. Indeed Γ acts transitively on $\mathcal{E}^\pm$ (resp. $\mathcal{V}^\pm$) and the stabilizer of $e_* \in \mathcal{E}$ (resp. $v_* \in \mathcal{V}$) is $\Gamma_{e_*} = \Gamma_0(pN^+)$ (resp. $\Gamma_0(N^+)$).

By *extended norm* on a space A we mean a function $\|\cdot\|: A \to \mathbb{R}_{\geq 0} \cup \{+\infty\}$ satisfying the usual properties of a norm, extended in a natural way to the semigroup of values $\mathbb{R}_{\geq 0} \cup \{+\infty\}$. Let $|\cdot|$ denote the absolute value of K_p. Let $|\cdot|_{L_*}$ and $|\cdot|_{\widehat{L}_*}$ be two norms on on $\mathbf{P}_n(K_p)$. We require the first one to be $\mathrm{GL}_2(L_*) = \mathrm{GL}_2(\mathbb{Z}_p)$ -invariant and the second one to be $\mathrm{GL}_2(\widehat{L}_*)$ -invariant. Choose also a $\mathrm{GL}_2(L_*) \cap \mathrm{GL}_2(\widehat{L}_*)$-invariant norm $|\cdot|_{L_*,\widehat{L}_*}$ on $\mathbf{P}_n(K_p)$. We may choose, for example, $|\cdot|_{L_*,\widehat{L}_*} = |\cdot|_{L_*} := |\cdot|$ to be the supremum of the absolute values of the coefficients of a polynomial and set $|\cdot|_{\widehat{L}_*} := \left|\omega_p^{-1}\cdot\right|_{L_*}$, where $\omega_p \in \mathcal{R}^\times$ is an element normalizing Γ such that $\widehat{v}_* = \omega_p v_*$. By duality we can consider the corresponding norms on $\mathbf{V}_n(K_p)$. Set $L_+ := L_*$, $L_- := \widehat{L}_*$ and define extended norms on $C(\mathcal{V}, \mathbf{V}_n(K_p))$ and $C_0(\mathcal{E}, \mathbf{V}_n(K_p))$ by the rules

$$\|c\| := \max\left\{\|c\|_+, \|c\|_-\right\},\ \|c\|_\pm := \sup_{v\in\mathcal{V}^\pm}|\gamma_v \cdot c(v)|_{L_\pm},$$
$$\|c\| := \sup_{e\in\mathcal{E}^+}|\gamma_e \cdot c(e)|.$$

Here, γ_v (resp. γ_e) is any element in $\mathrm{GL}_2^+(\mathbb{Q}_p)$ such that $\gamma_v(v) = v_\pm$ (resp. $\gamma_e(e) = e_*$), $v_\pm = [L_\pm]$ and $\mathrm{GL}_2^+(\mathbb{Q}_p) \subset \mathrm{GL}_2(\mathbb{Q}_p)$ is the subgroup of those elements g such that $\mathrm{ord}_p(\det(g)) \equiv 0 \bmod 2$. The invariance properties of the above norms imply that the above definitions do not depend on the choice of the sets $\{\gamma_v\}$ and $\{\gamma_e\}$. We view $C_{har}(\mathcal{E}, \mathbf{V}_n(K_p)) \subset C_0(\mathcal{E}, \mathbf{V}_n(K_p))$ with the extended norm obtained from $C_0(\mathcal{E}, \mathbf{V}_n(K_p))$.

Let $C(K_p)$ denote either $C(\mathcal{V}, \mathbf{V}_n(K_p)))$, $C_0(\mathcal{E}, \mathbf{V}_n(K_p))$ or $C_{har}(\mathcal{E}, \mathbf{V}_n(K_p))$ and write $C^b(K_p) := \{c \in C : \|c\| < \infty\}$. In all these cases the restriction of $\|\cdot\|$ to $C^b(K_p)$ is a norm with respect to which $C^b(K_p)$ is a Banach space over K_p. One has the following refinement of (4).

Proposition 5. *For every* even *integer $n \geq 0$, the natural sequence*

$$0 \to C^b_{har}(\mathcal{E}, \mathbf{V}_n(K_p)) \to C^b_0(\mathcal{E}, \mathbf{V}_n(K_p)) \to C^b(\mathcal{V}, \mathbf{V}_n(K_p)) \to 0$$

is an exact sequence of $\mathrm{PGL}_2(\mathbb{Q}_p)$*-modules.*

Proof. As explained in [RS, §2.6], the result can be viewed as a reformulation of [T2, pag. 564-566]. ☐

Quite generally, let us assume that Γ is a group acting (from the left) on a set $\mathcal{S}$ and that $\Gamma\backslash\mathcal{S}$ is finite. Write $\mathcal{S}=\bigsqcup_{i\in I}\mathcal{S}_i$, with $\mathcal{S}_i=\Gamma s_i$. Let $\Gamma_i\subset\Gamma$ be the stabilizer of s_i in Γ and let $\{\gamma_s^i\}_{s\in\mathcal{S}_i}$ be a set of representatives for the coset space $\Gamma_i\backslash\Gamma$ such that $\gamma_s^i s=s_i$ for all $s\in\mathcal{S}_i$. Let A be Γ-module endowed with Γ_i-invariant non-archimedean norms $|-|_i$. On the group of functions $C(\mathcal{S},A)$, define an extended norm by the rule

$$\|c\|:=\sup_{i\in I,s\in\mathcal{S}_i}\left|\gamma_s^i c\left(s\right)\right|_i=\sup_{i\in I,s\in\mathcal{S}_i}\left|\left(\gamma_s^i c\right)\left(s_i\right)\right|_i. \tag{7}$$

Let $C^b\left(\mathcal{S},A\right)$ be the subgroup of those elements $c\in C(\mathcal{S},A)$ such that $\|c\|<+\infty$. Note that, since the norm $|-|_i$ on A is Γ_i-invariant, $\|-\|_i:=\sup\limits_{s\in\mathcal{S}_i}\left|\gamma_s^i c\left(s\right)\right|_i$ does not depend on the choice of the set of representatives $\{\gamma_s^i\}_{s\in\mathcal{S}_i}$ and, furthermore, $\|-\|_i$ is a Γ-invariant function on $C\left(\mathcal{S},A\right)$. It follows that $\|-\|=\sup\limits_i\|-\|_i$ does not depend on the choice of the set $\{\gamma_s^i\}_{i\in I,s\in\mathcal{S}_i}$ and $\|-\|$ is a Γ-invariant extended norm.on $C(\mathcal{S},A)$. In particular, the definition of $C^b\left(\mathcal{S},A\right)$ does not depend on the choice of the set of representatives $\{\gamma_s^i\}_{i\in I,s\in\mathcal{S}_i}$ and $C^b\left(\mathcal{S},A\right)$ is a Γ-submodule of $C\left(\mathcal{S},A\right)$, endowed with a Γ-invariant norm. It is also clear from this discussion that, if we write $C\left(\mathcal{S},A\right)=\bigoplus_{i\in I}C\left(\mathcal{S}_i,A\right)$, $C^b\left(\mathcal{S},A\right)=\bigoplus_{i\in I}C^b\left(\mathcal{S}_i,A\right)$. We finally remark that, choosing equivalent Γ_i-invariant norms $|-|_i'\sim|-|_i$ on A yields $\|-\|_i'\sim\|-\|_i$ and the definition of $C^b\left(\mathcal{S},A\right)$ is unchanged.

Suppose now that we have given $\Gamma\subset G$, with G acting on $\mathcal{S}$. Then we have $\tau\colon\Gamma\backslash\mathcal{S}\twoheadrightarrow G\backslash\mathcal{S}$ and we write $\mathcal{S}=\bigsqcup_{j\in J}\mathcal{S}_j$, with $\mathcal{S}_j=Gs_j$. For every j, $\mathcal{S}_j=\bigsqcup_{i:\tau(\mathcal{S}_i)=\mathcal{S}_j}\mathcal{S}_i$, so that $I=\bigsqcup_{j\in J}I_j$ where $I_j:=\{i:\tau\left(\mathcal{S}_i\right)=\mathcal{S}_j\}$. Write $G=\bigsqcup_{k\in K_j}G_jg_k$. For every $i\in I_j$ there exists a unique $k_i\in K_j$ such that $g_{k_i}s_i=s_j$. Suppose we have choosen $\{\gamma_s^i\}_{s\in\mathcal{S}_i}$ such that $\gamma_s^i s=s_i$ for all $s\in\mathcal{S}_i$. Let $s\in\mathcal{S}_j$, say $s\in\mathcal{S}_i$ with $i\in I_j$, and set $g_s^j:=g_{k_i}\gamma_s^i$ for the unique $k_i\in K_j$ such that $g_{k_i}s_i=s_j$. Then $g_s^j s=g_{k_i}\gamma_s^i s=g_{k_i}s_i=s_j$, so that $\{g_s^j\}_{s\in\mathcal{S}_j}$ is a family of representatives for the coset space $G_j\backslash G$ such that $g_s^j s=s_j$. Suppose that we have given G_j-invariant norms $|-|_j$ on A. Then, for every $i\in I_j$, the norm $|-|_i:=|g_{k_i}-|_j$ is Γ_i-invariant. Indeed, suppose $\gamma s_i=s_i$, which is equivalent to $\gamma g_{k_i}^{-1}s_j=g_{k_i}^{-1}s_j$, i.e. $g_{k_i}\gamma g_{k_i}^{-1}s_j=s_j$. Then $g_{k_i}\gamma g_{k_i}^{-1}\in G_j$, so that

$$|\gamma a|_i:=|g_{k_i}\gamma a|_j=\left|g_{k_i}\gamma g_{k_i}^{-1}g_{k_i}a\right|_j=|g_{k_i}a|_j=:|a|_i.$$

We finally remark that we have, for $c\in C\left(\mathcal{S},A\right)$,

$$\begin{aligned}\|c\|_j&:=\sup_{s\in\mathcal{S}_j}\left|g_s^j c\left(s\right)\right|_j=\sup_{i\in I_j,s\in\mathcal{S}_i}\left|g_{k_i}\gamma_s^i c\left(s\right)\right|_j=\sup_{i\in I_j,s\in\mathcal{S}_i}\left|\gamma_s^i c\left(s\right)\right|_i\\&=\sup_{i\in I_j}\|c\|_i.\end{aligned}$$

Since I_j is finite we see that $\|c\|_j<+\infty$ if and only if $\|c\|_i<+\infty$ for every $i\in I_j$. Taking the sup over all js we see that $C^b\left(\mathcal{S},A\right)$ does not depend on the choice of G such that $\Gamma\subset G$, once the choice of the above norms $|-|_i:=|g_{k_i}-|_j$ for $i\in I_j$ has been made. In particular, the following remark holds.

Remark 6. *Suppose that A is a Γ-module, let $\mathcal{S}=\mathcal{E}^+$ or $\mathcal{V}$ and consider the inclusion $\Gamma\subset G:=\mathrm{GL}_2^+(\mathbb{Q}_p)$. Write $\mathcal{S}=\bigsqcup_{i\in I}\mathcal{S}_i$, with $\mathcal{S}_i=\Gamma s_i$ and define $\|c\|$ by (7). Then $C^b\left(\mathcal{S},A\right)$ as defined before Proposition 5 is $\{c\in C\left(\mathcal{S},A\right):\|c\|<+\infty\}$.*

The following proposition is the key result in proving the existence of p-adic integration theories in both the definite and the indefinite setting. Since $C\left(\mathcal{S},A\right)=\bigoplus_{i\in I}C\left(\mathcal{S}_i,A\right)$,

$C^b(\mathcal{S},A)=\bigoplus_{i\in I}C^b(\mathcal{S}_i,A)$ and Γ acts transitively on $\mathcal{S}_i$, it readly follows from [RS, Proposition 3.7].

Proposition 7. *Suppose that Γ is a finitely generated group. For $i=0,1$ the inclusion $\iota: C^b(\mathcal{S},A)\subset C(\mathcal{S},A)$ induces an isomorphism*

$$\iota\colon H^i(\Gamma,C^b(\mathcal{S},A))\xrightarrow{\simeq}H^i(\Gamma,C(\mathcal{S},A)).$$

3.2. **The existence of p-adic integration theories.** Let $\mathcal{A}_n(\mathbb{P}^1(\mathbb{Q}_p),K_p)$ be the space of K_p-valued locally analytic functions on $\mathbb{P}^1(\mathbb{Q}_p)$ with a pole of order at most n at ∞, and let $\mathcal{D}_n(\mathbb{P}^1(\mathbb{Q}_p),K_p)$ be its strong K_p-dual. The space $\mathcal{A}_n(\mathbb{P}^1(\mathbb{Q}_p),K_p)$ carries a right action of $\mathrm{GL}_2(\mathbb{Q}_p)$ defined by the rule $f\cdot\gamma=\frac{(cx+d)^n}{\det(\gamma)^{n/2}}\cdot f(\frac{ax+b}{cx+d})$, for any $f\in\mathcal{A}_n(\mathbb{P}^1(\mathbb{Q}_p),K_p)$ and $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{GL}_2(\mathbb{Q}_p)$. Note that $\mathbf{P}_n(K_p)$ is a natural $\mathrm{GL}_2(\mathbb{Q}_p)$-submodule of it. We define a left $\mathrm{GL}_2(\mathbb{Q}_p)$-action on $\mathcal{D}_n(\mathbb{P}^1(\mathbb{Q}_p),K_p)$ and $\mathbf{V}_n(K_p)$ by the rule $(\gamma\mu)(f):=\mu(\gamma f)$. The $\mathrm{GL}_2(\mathbb{Q}_p)$-module $\mathcal{D}^0_n(\mathbb{P}^1(\mathbb{Q}_p),K_p)$ is defined by the following exact sequence:

$$0\to\mathcal{D}^0_n(\mathbb{P}^1(\mathbb{Q}_p),K_p)\to\mathcal{D}_n(\mathbb{P}^1(\mathbb{Q}_p),K_p)\to\mathbf{V}_n(K_p)\to 0,$$

i.e. is the space of locally analytic distributions that are zero on the polynomials of degree $\leq n$.

Definition 8. *$\mathcal{D}^0_n(\mathbb{P}^1(\mathbb{Q}_p),K_p)^b\subset\mathcal{D}_n(\mathbb{P}^1(\mathbb{Q}_p),K_p)$ is the sub $\mathrm{GL}_2(\mathbb{Q}_p)$-module of distributions $\mu\in\mathcal{D}^0_n(\mathbb{P}^1(\mathbb{Q}_p),K_p)$ for which there is a constant A such that, for all $i\geq 0$, $j\geq 0$, and all $a\in\mathbb{Z}_p$,*

$$|\mu((x-a)^i|a+p^j\mathbb{Z}_p)|\leq p^{A-j(i-1-k/2)}.$$

There is a natural morphism of $\mathrm{GL}_2(\mathbb{Q}_p)$-modules

$$r\colon\mathcal{D}^0_n(\mathbb{P}^1(\mathbb{Q}_p),K_p)\twoheadrightarrow C_{har}(\mathcal{E},\mathbf{V}_n(K_p)),$$
$$r(\mu)(e)(P)=\int_{U_e}P(t)d\mu(t):=\mu(P\cdot\chi_{U_e}).$$

Here $U_e\subset\mathbb{P}^1(\mathbb{Q}_p)$ is the open compact subset of $\mathbb{P}^1(\mathbb{Q}_p)$ corresponding to the ends leaving from the oriented edge e, and χ_{U_e} stands for its characteristic function. By [DT, Theorem 2.3.2], it restricts to an isomorphism

$$r\colon\mathcal{D}^0_n(\mathbb{P}^1(\mathbb{Q}_p),K_p)^b\xrightarrow{\sim}C^b_{har}(\mathcal{E},\mathbf{V}_n(K_p)),\tag{8}$$

which by abuse of notation we denote with the same symbol r. The same abuse will be made for the several maps that r induces in cohomology, as below.

The following theorem is [T1, Prop. 9] (definite setting) and [RS, Theorem 3.5] (indefinite setting). Remark 6 allow us to recast the definite case in the framework of the techniques employed in [RS]. As explained in [RS, §3.1], once (8) is given, the proof is an application of Proposition 5 and Proposition 7.

Theorem 9. *For $i=0,1$, $r\colon H^i(\Gamma,\mathcal{D}^0_n(\mathbb{P}^1(\mathbb{Q}_p),K_p)^b)\xrightarrow{\sim}H^i(\Gamma,C_{har}(\mathbf{V}_n(K_p)))$ is an Hecke equivariant isomorphism.*

4. Monodromy modules and p-adic Abel-Jacobi maps

Let $k_p:=K_p\cap\mathbb{Q}_p^{ur}$ be the maximal unramified subextension of K_p. For the rest of the paper we write $\mathrm{Div}(\mathcal{H}_p)$ (resp. $\mathrm{Div}^0(\mathcal{H}_p)$) to denote the group of divisors (resp. degree zero divisors) supported on $\mathbb{Q}_p^{ur}-\mathbb{Q}_p$ that are fixed by the action of $G_{\mathbb{Q}_p^{ur}/k_p}$. Of course, in our applications to the quadratic field K considered in the introduction and §5, we have $k_p=K_p$. Having fixed our field K_p, we simply write $\mathbf{P}_n=\mathbf{P}_n(K_p)$, $\mathbf{V}_n=\mathbf{V}_n(K_p)$, $\mathcal{D}^0_n(\mathbb{P}^1(\mathbb{Q}_p)):=\mathcal{D}^0_n(\mathbb{P}^1(\mathbb{Q}_p),K_p)$.and $\mathcal{D}^0_n(\mathbb{P}^1(\mathbb{Q}_p))^b:=\mathcal{D}^0_n(\mathbb{P}^1(\mathbb{Q}_p),K_p)^b$.

Definition 10. *Define pairings*

$$\begin{array}{rcll} \int_{-}^{-} - \omega_{-}^{\log}\colon & \mathrm{Div}^0(\mathcal{H}_p) \otimes \mathbf{P}_n \otimes \mathcal{D}_n^0(\mathbb{P}^1(\mathbb{Q}_p)) & \rightarrow & K_p \\ & (\tau_2 - \tau_1) \otimes P \otimes \mu & \mapsto & \int_{\tau_1}^{\tau_2} P\,\omega_{\mu}^{\log} \end{array}$$

$$\begin{array}{rcll} \int_{-}^{-} - \omega_{-}^{\mathrm{ord}}\colon & \mathrm{Div}^0(\mathcal{H}_p) \otimes \mathbf{P}_n \otimes \mathcal{D}_n^0(\mathbb{P}^1(\mathbb{Q}_p)) & \rightarrow & K_p \\ & (\tau_2 - \tau_1) \otimes P \otimes \mu & \mapsto & \int_{\tau_1}^{\tau_2} P\omega_{\mu}^{\mathrm{ord}} \end{array}$$

where for any τ_1, $\tau_2 \in \mathcal{H}_p$, $P \in \mathbf{P}_n$ and $\mu \in \mathcal{D}_n^0(\mathbb{P}^1(\mathbb{Q}_p))$:

$$\int_{\tau_1}^{\tau_2} P\,\omega_{\mu}^{\log} := \int_{\mathbb{P}^1(\mathbb{Q}_p)} \log_p\left(\frac{t-\tau_2}{t-\tau_1}\right) P(t)d\mu(t), \quad \int_{\tau_1}^{\tau_2} P\omega_{\mu}^{\mathrm{ord}} := \sum_{e:red(\tau_1)\rightarrow red(\tau_2)} \int_{U_e} P(t)d\mu(t).$$

As explained in [RS, §3.2], the above pairings are $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant. By Theorem 9,

$$H^{i_{N^-}}(\Gamma, \mathcal{D}_n^0(\mathbb{P}^1(\mathbb{Q}_p))^b) \xrightarrow{\sim} H^{i_{N^-}}(\Gamma, C_{har}(\mathcal{E}, \mathbf{V}_n))) =: \mathbf{H}\left(K_p\right).$$

In particular, by means of this identification and by cap product, the above pairings yield

$$\Psi^{\log}, \Psi^{\mathrm{ord}}\colon \quad H_1(\Gamma, \mathrm{Div}^0(\mathcal{H}_p) \otimes \mathbf{P}_n) \longrightarrow \mathbf{H}(K_p)^{\vee}. \tag{9}$$

Let

$$H_{i_{N^-}+1}\left(\Gamma, \mathbf{P}_n\right) \xrightarrow{\partial} H_{i_{N^-}}\left(\Gamma, \mathrm{Div}^0\left(\mathcal{H}_p\right) \otimes \mathbf{P}_n\right) \xrightarrow{i} H_{i_{N^-}}\left(\Gamma, \mathrm{Div}\left(\mathcal{H}_p\right) \otimes \mathbf{P}_n\right) \tag{10}$$

be the exact sequence induced by the exact sequence

$$0 \rightarrow \mathrm{Div}^0\left(\mathcal{H}_p\right) \otimes \mathbf{P}_n \rightarrow \mathrm{Div}\left(\mathcal{H}_p\right) \otimes \mathbf{P}_n \rightarrow \mathbf{P}_n \rightarrow 0.$$

The following Theorem, which is well known in the definite setting, is proved in [RS, Theorem 3.11] in the indefinite setting. In order to emphasize the analogy between the definite and the indefinite setting, we explain how the proof offered in [RS, Theorem 3.11] in the indefinite setting works in the definite setting too. Let pr_c be the projection onto the cuspidal part.

Theorem 11. *For every $n \geq 0$ the morphism*

$$\mathrm{pr}_c \circ \Psi^{\mathrm{ord}} \circ \partial\colon H_{i_{N^-}+1}(\Gamma, \mathbf{P}_n) \rightarrow \mathbf{H}(K_p)_c^{\vee}$$

is surjective and induces an Hecke equivariant isomorphism

$$(\Psi^{\mathrm{ord}} \circ \partial)_c\colon H_{i_{N^-}+1}(\Gamma, \mathbf{P}_n)^c \xrightarrow{\simeq} \mathbf{H}(K_p)_c^{\vee}.$$

Proof. Let us assume that we are in the definite setting (so that $\mathbf{H}(K_p) = \mathbf{H}(K_p)_c$). As in [RS, proof of Theorem 3.11] one can produce a commutative diagram

$$\begin{array}{ccccc} H_1(\Gamma, \mathbf{P}_n) & \xrightarrow{\partial} & H_0(\Gamma, \mathrm{Div}^0(\mathcal{H}_p) \otimes \mathbf{P}_n) & & \\ \downarrow & & \downarrow \Psi_3 \circ \Psi_2 \circ \Psi_1 & & \\ H^1(\Gamma, \mathbf{V}_n)^{\vee} & \xrightarrow{\delta^{\vee}} & H^0(\Gamma, C_0(\mathcal{E}, \mathbf{V}_n))^{\vee} & \xrightarrow{\mathrm{pr}_c \circ \Psi_4} & \mathbf{H}(K_p)_c^{\vee}, \end{array}$$

where $\Psi^{\mathrm{ord}} = \Psi_4 \circ \Psi_3 \circ \Psi_2 \circ \Psi_1$, the left vertical arrow is an isomorphism and $\delta^{\vee}$ is the dual of δ arising from the first boundary map of the second line of (5) with $A = \mathbf{V}_n\left(K_p\right)$. It remains to prove that $\mathrm{pr}_c \circ \Psi_4 \circ \delta^{\vee} = \Psi_4 \circ \delta^{\vee}$ is an isomorphism. Equivalently, since the composition $\mathrm{pr}_c \circ \Psi_4 = \Psi_4$ is dual to the morphism

$$\mathbf{H}(K_p)_c = \mathbf{H}(K_p) \rightarrow H^1(\Gamma, C_0(\mathcal{E}, \mathbf{V}_n)),$$

we need to show that the morphism

$$\mathbf{H}(K_p) = H^0(\Gamma, C_{har}(\mathbf{V}_n)) \rightarrow H^0(\Gamma, C_0(\mathcal{E}, \mathbf{V}_n)) \rightarrow H^1(\Gamma, \mathbf{V}_n) = H_1(\Gamma, \mathbf{P}_n)^{\vee}$$

induces an isomorphism. This is the content of Proposition 2, that replaces Proposition 3 used in [RS, proof of Theorem 3.11] in the indefinite setting. $\square$

Set $\mathbb{T}_p := \mathbb{T} \otimes_{\mathbb{Q}} \mathbb{Q}_p$. As explained in [RS, Corollary 3.7], Theorem 11 has the following important consequence.

Corollary 12. *There exists a unique endomorphism $\mathcal{L} = \mathcal{L}_{N^-} \in \mathbb{T}_p$ such that*

$$\mathrm{pr}_c \circ \Psi^{\log} \circ \partial = \mathcal{L} \circ \ \mathrm{pr}_c \circ \Psi^{\mathrm{ord}} \circ \partial : H_2(\Gamma, \mathbf{P}_n) \to \mathbf{H}(K_p)_c^{\vee}.$$

Remark 13. *By means of (6) we may attach several a priori different $\mathcal{L}$-invariants $\mathcal{L}_{N^-}(f)$ to a modular form $f \in S_k^1(\Gamma_0(N), \mathbb{C})^{pN^- - new}$ and a factorization $N = pN^+N^-$. Furthermore, in the indefinite setting, i.e. when N^- is divisible by an even number of primes, we may consider $\mathcal{L}_{N^-}^{w_\infty}(f)$ where $w_\infty \in \{\pm 1\}$ is the choice of a sign of W_∞. As conjectured in* [RS] *(in the indefinite setting) and proved in* [Se3] *all these $\mathcal{L}$-invariants are equal (see also the subsequent Theorem 16). Thanks to* [IS] *or* [BDI] *(where f is assumed to be split at p) this is also true in the definite setting.*

4.1. **Monodromy modules.** We follow [RS, §4] and explain how we may attach to the space of weight k-modular forms on $\mathcal{B}$ two monodromy modules. The first construction, which is well known, is obtained by means of Fontaine theory: we refer the reader to [J], [Sc], [IS, §10.1] and [CI] for details. The second construction is obtained as a by-product of the p-adic integration theory developed there: the reference is [RS, §4], which also contains a brief account on the first construction and the connection between them.

Write $\sigma \in \mathrm{Aut}(k_p)$ for the absolute Frobenius of k_p. Let $\mathbb{T}_p$ be a finite dimensional commutative $\mathbb{Q}_p$-algebra and write $\mathbb{T}_{k_p} = \mathbb{T}_p \otimes_{\mathbb{Q}_p} k_p$ and $\mathbb{T}_{K_p} = \mathbb{T}_p \otimes_{\mathbb{Q}_p} K_p$. Set $\sigma_{\mathbb{T}_{k_p}} :=$ $\mathrm{Id} \otimes \sigma$ on $\mathbb{T}_{k_p}$.

Definition 14. *A* two dimensional monodromy $\mathbb{T}_p$-module over K_p *is a 4-ple $(D, \varphi, N, F^{\cdot})$ where D is a $\mathbb{T}_{k_p}$-module, $\varphi \colon D \to D$ is σ-linear endomorphism (i.e. $\varphi(ax) = \sigma(a)x$ for all $a \in k_p, x \in D$) and $N \colon D \to D$ is a k_p-linear endomorphism such that*

(a) *$F^{\cdot}$ is a filtration on the K_p-vector space $D \otimes_{k_p} K_p$ of the form*

$$D \otimes K_p = F^0 \supset F^1 = \cdots = F^{k-1} \supset F^k = 0$$

where F^{k-1} is a free $\mathbb{T}_{K_p}$-module of rank one;

(b) *$D \otimes K_p = F^{k-1} \oplus N_{K_p}(D \otimes K_p)$ as a $\mathbb{T}_{K_p}$-module, with $N_{K_p} \colon F^{k-1} \to N_{K_p}(D \otimes K_p)$ a $\mathbb{T}_{K_p}$-module isomorphism.*

(c) *$N \cdot \varphi = p\varphi \cdot N$ and, for any $T \in \mathbb{T}_{k_p}$, $\varphi T = \sigma_{\mathcal{H}_{k_p}}(T)\varphi$ and $TN = NT$.*

Consider the slope decomposition $D = \bigoplus_{\alpha \in \mathbb{Q}} D^{\alpha}$. Since $N \neq 0$ and $N(D^{\alpha+1}) \subset D^{\alpha}$, there exists $\lambda \in \mathbb{Q}$ such that $D^{\lambda}, D^{\lambda+1} \neq 0$ are free $\mathbb{T}_{k_p}$-modules of rank 1 and the map $N \colon D^{\lambda+1} \to D^{\lambda}$ is non-zero. Attached to a monodromy module there is a so called Fontaine-Mazur $\mathcal{L}$-invariant.

Definition 15. *The* $\mathcal{L}$-invariant $\mathcal{L}_D$ of $D = (D, \varphi, N, F^{\cdot})$ *is defined to be the unique element $\mathcal{L}_D \in \mathbb{T}_{K_p}$ such that*

$$x - \mathcal{L}_D N_{K_p}(x) \in F^{k-1} \text{ for every } x \in D^{\lambda+1} \otimes K_p.$$

4.1.1. *Monodromy modules arising from Fontaine theory.* We are going to recall how to attach to the space of weight k cusp forms on our (definite or indefinite) quaternion algebra $\mathcal{B}$ a monodromy module by means of Fontaine theory.

The definite setting. Let $\mathcal{B}_{pN^-}$ be the indefinite quaternion algebra ramified at the primes dividing pN^-, let $\mathcal{R}_0^{pN^-}(N^+)$ be an Eichler order of level N^+, defined similarly as in §2 with pN^- in place of N^-, and let $\Gamma_0^{pN^-}(N^+) := \mathcal{R}_0^{pN^-}(N^+)_1^\times$ be the subgroup of norm one elements in $\mathcal{R}_0^{pN^-}(N^+)^\times$. There is a Shimura curve $X = X_{N^+,pN^-}$ over $\mathbb{Q}$, whose $\mathbb{C}$-points are $X(\mathbb{C}) = \Gamma_0^{pN^-}(N^+)\backslash\mathcal{H}$. Let Γ be the arithmetic group defined in §2 with the choice of the definite quaternion algebra $\mathcal{B} = \mathcal{B}_{N^-}$ ramified at $N^-\infty$. The Cerednik-Drinfeld Theorem asserts that there is a rigid analytic isomorphism, defined over the quadratic unramified extension $\mathbb{Q}_{p^2}$ of $\mathbb{Q}_p$

$$\Gamma\backslash\mathcal{H}_p = X^{an},$$

where X^{an} is the rigid analytification of X.

In [IS, §10] it is offered a construction of a Chow motive $\mathcal{M}_n$, attached to the weight k modular forms on X. This motive is defined over $\mathbb{Q}$ and, for any field extension $H/\mathbb{Q}$, we let $\mathcal{M}_{n,H}$ be its base change to H. The Hecke algebra $\mathbb{T}_p$ acts on it and, after taking its p-adic realization $V := H_p\left(\mathcal{M}_{n,\overline{\mathbb{Q}}}, \mathbb{Q}_p\right)$, we can consider the filtered Frobenius module $\mathbb{D} = \mathbb{D}_{N^-} := \mathbb{D}_{st}(V)$ attached to (the restriction to a decomposition group at p) of V. As it is well known it is indeed a two dimensional monodromy $\mathbb{T}_p$-module over $\mathbb{Q}_p$ (see [IS]).
The indefinite setting. Let $\mathcal{B} = \mathcal{B}_{N^-}$ be the indefinite quaternion algebra ramified at the primes dividing N^-, let $\mathcal{R}_0^{N^-}(pN^+)$ be the Eichler order of level pN^+ defined in §2 and let $\Gamma_0^{N^-}(pN^+) := \mathcal{R}_0^{N^-}(pN^+)_1^\times$ be the subgroup of norm one elements in $\mathcal{R}_0^{N^-}(pN^+)^\times$. Again there is a Shimura curve $X = X_{pN^+,N^-}$ over $\mathbb{Q}$, whose $\mathbb{C}$-points are $X(\mathbb{C}) = \Gamma_0^{N^-}(pN^+)\backslash\mathcal{H}$.

As in [IS, §10] (see also [CI]) there is a Chow motive $\mathcal{M}_n$, attached to the weight k modular forms on X. This motive is defined over $\mathbb{Q}$ and, for any field extension $H/\mathbb{Q}$, we let $\mathcal{M}_{n,H}$ be its base change to H. Let $H_p\left(\mathcal{M}_{n,\overline{\mathbb{Q}}}, \mathbb{Q}_p\right)$ be its p-adic realization and let $V := H_p\left(\mathcal{M}_{n,\overline{\mathbb{Q}}}, \mathbb{Q}_p\right)^{p-new}$ be its p-new part. The Hecke algebra $\mathbb{T}_p$ acts on it and we can consider the filtered Frobenius module $\mathbb{D} = \mathbb{D}_{N^-} := \mathbb{D}_{st}(V)$ attached to (the restriction to a decomposition group at p) of V. As it is well known it is indeed a two dimensional monodromy $\mathbb{T}_p$-module (see [CI]).

4.2. **Monodromy modules arising from p-adic integration theory.** When we are in the indefinite setting, fix a choice of a sign $w_\infty \in \{\pm 1\}$. By [RS, Remark 2.4], §2.1 and §2.2, $\mathbf{H}(\mathbb{Q}_p)_c^{\vee,w_\infty}$ is a free module of rank 1 over $\mathbb{T}_p$. Let $U_p \in \mathbb{T}_p$ be the Hecke operator at p and let $\mathcal{L} \in \mathbb{T}_p$ be the element provided by Corollary 12. Define the $\mathbb{Q}_p$-vector space

$$\mathbf{D} = \mathbf{D}_{N^-} := \mathbf{H}(\mathbb{Q}_p)_c^{\vee,w_\infty} \oplus \mathbf{H}(\mathbb{Q}_p)_c^{\vee,w_\infty}. \tag{11}$$

It is indeed a $\mathbb{T}_p$-monodromy module over $\mathbb{Q}_p$ with the filtration, Frobenius and monodromy operators described below.

- a filtration $\mathbf{D} = F^0 \supsetneq F^1 = \cdots = F^{k-1} \supsetneq F^k = 0$, where for all $1 \le j \le k-1$,
$$F^j = \{(-\mathcal{L}x, x) : x \in \mathbb{T}_p\};$$
- a Frobenius operator φ given by the rule
$$\varphi(x,y) := (\sigma_{\mathbb{T}_{kp}}(x)U, p\,\sigma_{\mathbb{T}_{kp}}(y)U);$$
- a monodromy operator N defined by the rule
$$N(x,y) = (y,0).$$

The following Theorem, that was formulated as a conjecture in [RS] in the indefinite setting, is proved in [IS] (definite setting) and [Se3] (indefinite setting):

Theorem 16. $\mathcal{L}_{N^-}^{w_\infty} = \mathcal{L}_{\mathbb{D}_{N^-}}$.

As explained in [RS, §4], Theorem 16 implies that there is an isomorphism of mondoromy modules over $\mathbb{Q}_p$

$$\varphi : \mathbf{D}_{N^-} \simeq \mathbb{D}_{N^-}. \tag{12}$$

4.3. **p-adic Able-Jacobi maps.** Let $\mathbf{D}_{K_p}$ be the base change to K_p of $\mathbf{D}$ (as monodromy modules). Note that the underlying filtered K_p-vector space is $\mathbf{H}(K_p)_c^{\vee,w_\infty,2}$. Define

$$\Psi := -\Psi^{\log} \oplus \Psi^{\mathrm{ord}} \colon H_1(\Gamma, \mathrm{Div}^0(\mathcal{H}_p) \otimes \mathbf{P}_n) \longrightarrow \mathbf{H}(K_p)^\vee \oplus \mathbf{H}(K_p)^\vee$$

and set

$$\Phi := -\Phi^{\log} \oplus \Phi^{\mathrm{ord}} \colon H_1(\Gamma, \mathrm{Div}^0(\mathcal{H}_p) \otimes \mathbf{P}_n) \to \mathbf{D} \otimes K_p \tag{13}$$

for the natural composition of the above map(s) with the projection onto $\mathbf{H}(K_p)_c^{\vee,w_\infty}$.

By definition of Φ, the free $\mathbb{T}_{K_p}$-submodule of rank one

$$F^1 = F^m = F^{k-1} := \{(-\mathcal{L}x, x) : x \in \mathbf{H}(K_p)^{c,\vee,w_\infty}\}$$

of $\mathbf{D} \otimes K_p$ is $F^m = \mathrm{Im}(\Phi \circ \partial)$.

The following Lemma is proved in [RS, Lemma 3.10] in the indefinite setting. In the definite setting, it is an easy consequence of the irreducibility of $\mathbf{P}_n$, when $n > 0$.

Lemma 17. *$H_{i_{N^-}}(\Gamma, \mathbf{P}_n(K_p)) = 0$ in the indefinite setting and in the definite settiing when $n > 0$.*

In particular, as in [RS], when Lemma 17 is in force, (10) yields

$$H_{i_{N^-}}(\Gamma, \mathrm{Div}(\mathcal{H}_p) \otimes \mathbf{P}_n) \simeq \frac{H_{i_{N^-}}(\Gamma, \mathrm{Div}^0(\mathcal{H}_p) \otimes \mathbf{P}_n)}{\mathrm{Im}\,\partial}. \tag{14}$$

Unfortunately, when $n = 0$ and we are in the definite setting, $H_0(\Gamma, K_p) = K_p$ is non-zero. However, for every prime $l \nmid N$, $t_l := T_l - (l+1)$ is zero on $H_0(\Gamma, K_p)$. It follows that, for every $z \in H_{i_{N^-}}(\Gamma, \mathrm{Div}(\mathcal{H}_p) \otimes \mathbf{P}_n)$, $t_l z = i\,(z')$ for a unique $z' \in \frac{H_{i_{N^-}}(\Gamma, \mathrm{Div}^0(\mathcal{H}_p) \otimes \mathbf{P}_n)}{\mathrm{Im}\,\partial}$. In other words there is

$$t_l \colon H_{i_{N^-}}(\Gamma, \mathrm{Div}(\mathcal{H}_p) \otimes \mathbf{P}_n) \to \frac{H_{i_{N^-}}(\Gamma, \mathrm{Div}^0(\mathcal{H}_p) \otimes \mathbf{P}_n)}{\mathrm{Im}\,\partial}. \tag{15}$$

Note also that t_l acts invertibly on $\mathbf{D}$.

Definition 18. *The p*-adic Abel-Jacobi maps *are the morphisms*

$$\Psi^{AJ} \colon H_1(\Gamma, \mathrm{Div}(\mathcal{H}_p) \otimes \mathbf{P}_n(K_p)) \longrightarrow \frac{\mathbf{H}(K_p)^\vee \oplus \mathbf{H}(K_p)^\vee}{\mathrm{Im}\,\Psi \circ \partial_2}$$

and

$$\Phi^{AJ} = \mathrm{pr}_c \circ \Psi^{AJ} \colon H_1(\Gamma, \mathrm{Div}(\mathcal{H}_p) \otimes \mathbf{P}_n(K_p)) \longrightarrow \mathbf{D} \otimes K_p / F^m. \tag{16}$$

induced respectively by Ψ and Φ, together with the isomorphism (14) when Lemma 17 is in force.

When or we are in the definite setting and $n = 0$ we use (15) and define

$$\Phi^{AJ} := t_l^{-1} \Phi t_l.$$

It is easily checked that, in the definite setting when $n = 0$, the definition does not depend on the choice of the auxiliary prime $l \nmid N$.

5. Heegner and Darmon cycles

Let $K/\mathbb{Q}$ be a quadratic field in which p remains inert. Assume for simplicity that the discriminant D_K of K is prime to N. As explained in the introduction, this induces a factorization of N as $N = pN^+N^-$, where $(N^+, N^-) = 1$ and all prime factors of N^+ (respectively N^-) split (resp. remain inert) in K.

Crucial for our construction is the following *Heegner hypothesis*, which we assume for the rest of this section.

Assumption. *N^- is the square-free product of an odd (resp. even) number of primes if $K/\mathbb{Q}$ is imaginary (resp. real).*

Let K_p denote the completion of K at p, a quadratic unramified extension of $\mathbb{Q}_p$. When $K/\mathbb{Q}$ is imaginary (respreal) let $\mathcal{B}$ be the definite (resp. indefinite) quaternion algebra of discriminant $N^-\infty$ (resp. N^-), $\mathcal{R}$ be a $\mathbb{Z}[1/p]$-Eichler order of level N^+ in $\mathcal{B}$ and Γ be the subgroup of $\mathcal{R}^\times$ of elements of reduced norm 1, as defined in §2.

Fix $\mathcal{B}_p = \mathrm{M}_2(\mathbb{Q}_p)$, which allows us to regard Γ as a subgroup of $\mathrm{SL}_2(\mathbb{Q}_p)$. Choose also embeddings $\sigma_p\colon K \to K_p$ and, when $K/\mathbb{Q}$ is real, $\sigma_\infty\colon K \to \mathbb{R}$, that we use to regard K as a subfield of K_p and, when $K/\mathbb{Q}$ is real, also as a subfield of $\mathbb{R}$. In particular, $D_K^{-\frac{k-2}{4}} \in K_p$ via σ_p.

Let us denote by $\mathrm{Emb}(K, \mathcal{B})$ the set of all the $\mathbb{Q}$-algebra embeddings of K into $\mathcal{B}$. Let $\mathcal{O} \subset K$ be a $\mathbb{Z}[1/p]$-order of conductor $c \geq 1$, $(c, ND_K) = 1$, and let $\mathrm{Emb}(\mathcal{O}, \mathcal{R})$ be the set of $\mathbb{Z}[1/p]$-optimal embeddings of $\mathcal{O}$ into $\mathcal{R}$. Attached to an embedding $\Psi \in \mathrm{Emb}(\mathcal{O}, \mathcal{R})$ there are the following data:

- the two fixed points $\tau_\Psi, \overline{\tau}_\Psi \in \mathcal{H}_p \cap K$ for the action of $\Psi(K_p^\times)$ on $\mathcal{H}_p \cap K$, labelled in such a way that the action of $K^\times$ on the tangent space at τ_Ψ is given by the character $z \mapsto z/\overline{z}$;
- the unique vertex $v_\Psi \in \mathcal{V}$ which is fixed for the action of $\Psi(K_p^\times)$ on $\mathcal{V}$: we have $v_\Psi = \mathrm{red}(\tau_\Psi) = \mathrm{red}(\overline{\tau}_\Psi)$;
- the unique polynomial up to sign P_Ψ in $\mathbf{P}_2$ which is fixed by the action of $\Psi\left(K_p^\times\right)$ on $\mathbf{P}_2$ and satisfies $\langle P_\Psi, P_\Psi \rangle_{\mathbf{P}_2} = -D_K/4$. We single out one by

$$P_\Psi := \mathrm{Tr}(\Psi(\sqrt{D_K}/2) \cdot \begin{pmatrix} X & -X^2 \\ 1 & -X \end{pmatrix}) \in \mathbf{P}_2;$$

- the stabilizer Γ_Ψ of Ψ in Γ, that is,

$$\Gamma_\Psi = \Psi(K^\times) \cap \Gamma = \Psi(\mathcal{O}_1^\times)$$

  where $\mathcal{O}_1^\times := \{\gamma \in \mathcal{O}^\times, \mathrm{n}(\gamma) = 1\}$, which is finite when $K/\mathbb{Q}$ is imaginary;
- when $K/\mathbb{Q}$ is real, the generator $\gamma_\Psi := \Psi(u)$ of $\Gamma_\Psi/\{\pm 1\} \simeq \mathbb{Z}$, where $u \in \mathcal{O}_1^\times$ is the unique generator of $\mathcal{O}_1^\times/\{\pm 1\}$ such that $\sigma_\infty(u) > 1$.

For each $\tau \in \mathcal{H}_p$, we say that τ has positive orientation at p if $\mathrm{red}(\tau) \in \mathcal{V}^+$. We write $\mathcal{H}_p^+$ to denote the set of positive oriented elements in $\mathcal{H}_p$. We say that $\Psi \in \mathrm{Emb}(\mathcal{O}, \mathcal{R})$ has positive orientation whenever $v_\Psi \in \mathcal{V}^+$, i.e. $\tau_\Psi, \overline{\tau}_\Psi \in \mathcal{H}_p^+ \cap K$. Put

$$\mathrm{Emb}(\mathcal{O}, \mathcal{R}) = \mathrm{Emb}_+(\mathcal{O}, \mathcal{R}) \sqcup \mathrm{Emb}_-(\mathcal{O}, \mathcal{R})$$

with the obvious meaning. The group Γ acts on $\mathrm{Emb}(\ \mathcal{O}, \mathcal{R})$ by conjugation, preserving orientations.

5.1. **Heegner cycles.** Suppose in this subsection that $K/\mathbb{Q}$ is imaginary and, hence, that we are in the definite setting.

Definition 19. *The* naive Heegner cycle *attached to an embedding* $\Psi \in \mathrm{Emb}(\mathcal{O}, \mathcal{R})$ *is*

$$y_\Psi := \tau_\Psi \otimes P_\Psi^m \in H_0(\Gamma, \mathrm{Div}(\mathcal{H}_p) \otimes \mathbf{P}_n)$$

and its Abel-Jacobi image is $s_\Psi := \Phi^{AJ}(y_\Psi) \in \mathbf{D} \otimes K_p/F^m$.

It is easily checked that the definition of y_Ψ does not depend on the choice of Ψ in its conjugacy class.

Let notations be as in §4.1.1. We first assume $n > 0$. Thanks to [IS, Lemma 10.1] we have $CH^{m+1}(\mathcal{M}_n) = CH_0^{m+1}(\mathcal{M}_n)$. Thus, the classical p-adic Abel-Jacobi map takes the following form

$$cl_{0,H}^{m+1} \colon CH^{m+1}(\mathcal{M}_{n,H}) \to H^1(H, V(m+1)),$$

Thanks to the work of Faltings (see [F1] and [F2]) there is a comparison isomorphism between the p-adic étale and the de Rham cohomology of $\mathcal{M}_{n,K_p}$. This interpretation is made explicit in [CI], where Faltings' definition is compared with the more explicit definition of Coleman (see [C]). More precisely, there is a functor $\mathbb{D}_{st}$ from the category of p-adic representations of $G_{\overline{K}_p/K_p}$ to the category $MF := MF_{K_p}(\phi, N)$ of filtered (ϕ, N)-modules, realizing this comparison. One has an identification in MF between the de Rham cohomology $H_{dR}\left(\mathcal{M}_{n,\overline{\mathbb{Q}}}, \mathbb{Q}_p\right)$ and $\mathbb{D} := \mathbb{D}_{st}\left(H_p\left(\mathcal{M}_{n,\overline{\mathbb{Q}}}, \mathbb{Q}_p\right)\right)$. The ext group $H_{st}^1(K_p, V(m+1))$ is explicitely computed in [IS]. After a base change from $\mathbb{Q}$ to K_p, there is an identification:

$$H_{st}^1(K_p, V(m+1)) = Ext_{MF}^1(K_p, \mathbb{D}(m+1)) = \frac{\mathbb{D}_{K_p}}{F^{m+1}\mathbb{D}_{K_p}}.$$

As explained in [IS, §7], when H is a number field, the classical p-adic Abel-Jacobi map $cl_{0,H}^{m+1}$ factors throught the semistable Bloch-Kato Selmer group $H_{st}^1(H, V(m+1)) \subset H^1(H, V(m+1))$. When $H = K_p$, cl_{0,K_p}^{m+1} factors through $H_{st}^1(K_p, V(m+1)) \subset H^1(K_p, V(m+1))$.

In particular, the p-adic étale Abel-Jacobi map cl_{0,K_p}^{m+1} takes the form:

$$\Phi_{geom}^{AJ} \colon CH^{m+1}\left(\mathcal{M}_{n,K_p}\right) \to \frac{\mathbb{D}_{K_p}}{F^{m+1}\mathbb{D}_{K_p}}.$$

When $n = 0$, the degree zero correspondence t_l gives $t_l \colon CH^1(\mathcal{M}_0) \to CH_0^1(\mathcal{M}_0)$. Since t_l acts invertibly on $\mathbb{D}_{K_p}$, we may define

$$\Phi_{geom}^{AJ} \colon CH^1\left(\mathcal{M}_{0,K_p}\right) \xrightarrow{t_l} CH_0^{m+1}\left(\mathcal{M}_{0,K_p}\right) \to \frac{\mathbb{D}_{K_p}}{F^{m+1}\mathbb{D}_{K_p}} \xrightarrow{t_l^{-1}} \frac{\mathbb{D}_{K_p}}{F^{m+1}\mathbb{D}_{K_p}}.$$

Let H/K be the ring class field which corresponds to the order $\mathcal{O}$. As we assume that p is inert in K, $\sigma_p \colon H \hookrightarrow K_p$. Attached to the embedding Ψ, there are so called Heegner cycles $y_\Psi^{He} \in CH^{m+1}(\mathcal{M}_{n,H})$. Of course there is a commutative diagram

$$\begin{array}{ccc} CH^{m+1}(\mathcal{M}_{n,H}) & \to & H_{st}^1(H, V(m+1)) \\ \sigma_p \downarrow & & \downarrow \sigma_p \\ CH^1\left(\mathcal{M}_{0,K_p}\right) & \to & H^1(K_p, V(m+1)), \end{array}$$

where the vertical arrows are the restriction morphisms induced by σ_p.

Recall the identification (12): as explained in [RS] a different choice of φ is induced by multiplication by an element of $\mathbb{T}_p^\times$. In particular the property of being in the image

$\sigma_p\left(H^1_{st}\left(H, V\left(m+1\right)\right)\right)$ does not depend on the choice of φ. The following Theorem is proved in [BD1, §4.3] when $n=0$. It generalization to the case $n>0$, based on the results of [IS], is proved in [Se1].

Theorem 20. *We may choose (12) in such a way that*

$$\Phi^{AJ}\left(y_\Psi\right)=\Phi^{AJ}_{geom}\left(\sigma_p\left(y^{He}_\Psi\right)\right).$$

5.2. **Darmon cycles.** Suppose in this subsection that $K/\mathbb{Q}$ is real and, hence, that we are in the indefinite setting.

The Γ_Ψ-module $K_p\cdot\tau_\Psi\otimes D_K^{-\frac{k-2}{4}}P^m_\Psi\subset\mathrm{Div}(\mathcal{H}_p)\otimes\mathbf{P}_n$ is endowed with the trivial Γ_Ψ-action. Hence, the choice of the generator γ_Ψ for the cyclic group Γ_Ψ allow us to fix an identification $K_p=H_1(\Gamma_\Psi,K_p\cdot\tau_\Psi\otimes D_K^{-\frac{k-2}{4}}P^m_\Psi)$. The inclusion $K_p\cdot\tau_\Psi\otimes D_K^{-\frac{k-2}{4}}P^m_\Psi\subset\mathrm{Div}(\mathcal{H}_p)\otimes\mathbf{P}_n$ then induces the *cycle class map*

$$cl_\Psi\colon K_p=H_1(\Gamma_\Psi,K_p\cdot\tau_\Psi\otimes D_K^{-\frac{k-2}{4}}P^m_\Psi)\to H_1(\Gamma,\mathrm{Div}(\mathcal{H}_p)\otimes\mathbf{P}_n).$$

Definition 21. *The* Darmon cycle *attached to an embedding* $\Psi\in\mathrm{Emb}(\ \mathcal{O},\mathcal{R})$ *is*

$$y_\Psi:=cl_\Psi(1)\in H_1(\Gamma,\mathrm{Div}(\mathcal{H}_p)\otimes\mathbf{P}_n)$$

and its Abel-Jacobi image is $s_\Psi:=\Phi^{AJ}\left(y_\Psi\right)\in\mathbf{D}\otimes K_p/F^m$.

As explained in [RS, §5] the definition of y_Ψ does not depend on the choice of Ψ in its conjugacy class.

Let notations be as in §4.1.1 and let H/K be the narrow ring class field which corresponds to the order $\mathcal{O}$. Again we may consider the Bloch-Kato semistable Selmer group $H^1_{st}\left(H,V\left(m+1\right)\right)$. Again we have $\sigma_p\colon H\hookrightarrow K_p$. Recall the identification (12): as explained, the property of being in the image $\sigma_p\left(H^1_{st}\left(H,V\left(m+1\right)\right)\right)$ does not depend on the choice of φ. The Bloch-Kato exponential induces in this setting an identification

$$\exp:\frac{\mathbb{D}_{K_p}}{F^{m+1}\mathbb{D}_{K_p}}=H^1_{st}\left(K_p,V\left(m+1\right)\right)$$

Generalizing conjectures formulated in [D] and guided by the analogy with Theorem 20, the following conjecture is formulated in [RS]:

Conjecture 22. $\exp\left(\varphi\left(\Phi^{AJ}\left(y_\Psi\right)\right)\right)\in\sigma_p\left(H^1_{st}\left(H,V\left(m+1\right)\right)\right)$.

We refer the reader to [BD2], [Se2] and [GSS] for theoretical evidences of the above conjecture, where suitable linear combinations of Darmon cycles are shown to arise from global cohomology classes.

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